Random walks and electrical networks 2

$G=(V,E)$ finite, connected graph

$c=(c_e)_{e\in E}$ positive reals (conductances)

$r_e = \frac{1}{c_e}$ resistance

$\pi(v) = \sum_{u:u\sim v} c_{uv}$.

Random walk: (X_n) taking values in V

$$\mathbb{P}(X_{n+1}=v \mid X_n = u) := \frac{c_{uv}}{\pi(u)}.$$

Reminder

$h: V \to \mathbb{R}$ is harmonic at $u \in V$ if

$$h(u) = \frac{1}{\pi(u)} \sum_{v:v\sim u} c_{uv} h(v)$$

That is $\mathbb{E}_u h(X_1) = h(u)$

start with $X_0 = u$

Voltage: Fix $a, z \in V$. A function $h: V \to \mathbb{R}$ that is harmonic at all $V\setminus\{a,z\}$ is called a voltage.

Lemma: For every $\alpha, \beta \in \mathbb{R}$ there exists a unique voltage h s.t. $h(a)=\alpha$, $h(z)=\beta$.

Also if $\alpha=0, \beta=1$ then this unique voltage is given by $h(v) = \mathbb{P}_v(\tau_z < \tau_a)$ where $\tau_x := \min\{n \geq 0 : X_n = x\}$.

Flow: A flow (from a to z) is a function $\theta : \vec{E} \to \mathbb{R}$ s.t. directed edges

i) Antisymmetry: $\theta(uv) = -\theta(vu)$, $uv \in \vec{E}$.

ii) Node law: $\forall u \in V \setminus \{a, z\}$, $\sum_{v : v \sim u} \theta(uv) = 0$.

Current flow: If h is a voltage then $\theta(uv) = c_{uv}(h(v) - h(u))$ is the current flow of h.

Kirchoff cycle law: A flow θ satisfies the cycle law if for any directed cycle $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m$ it holds that

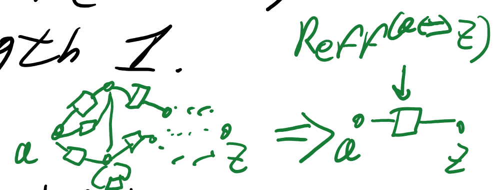
$$\sum_{i=1}^{m} r_{\vec{e}_i} \theta(\vec{e}_i) \qquad r_{\vec{e}_i} = r_{e_i}$$

Lemma: A flow is the current flow of a voltage iff it satisfies the cycle law. Then the voltage is unique up to an additive constant.

Strength: The strength of a flow θ is $\|\theta\| := \sum_{v : v \sim a} \theta(av) \underset{\text{Lemma}}{=} \sum_{u : u \sim z} \theta(uz)$

Lemma: There is a unique flow satisfying the cycle law with a given strength.

Def.: Unit current flow is the unique such flow with strength 1.



Effective resistance

Def: The ratio $\frac{h(z) - h(a)}{\|\theta\|}$

Lemma: The ratio $\frac{h(z)-h(a)}{\|\theta\|}$ is the same for all non-constant h and θ the current flow of h, and is positive.

Def.: This constant is the effective resistance of the network and denoted $R_{eff}(a \leftrightarrow z) = R_{eff}(a \leftrightarrow z; G, (r_e))$.

Effective conductance is the reciprocal

Proof: Let h_1, h_2 be non-constant voltages. Let θ_1, θ_2 be their current flows. Notice that $\|\theta_i\| \neq 0$ (e.g., by uniqueness of the current flow for a given strength). Normalize $\bar{h}_i := \frac{h_i}{\|\theta_i\|}$, so that the current flow $\bar{\theta}_i$ has strength 1. Uniqueness now gives that $\bar{\theta}_1 = \bar{\theta}_2$ and $\exists c$ s.t. $\bar{h}_1 = \bar{h}_2 + c$.

In particular,

$$\frac{h_1(z)-h_1(a)}{\|\theta_1\|} = \bar{h}_1(z) - \bar{h}_1(a) = \bar{h}_2(z) - \bar{h}_2(a) = \frac{h_2(z)-h_2(a)}{\|\theta_2\|}.$$

The ratio is positive, e.g., by considering the voltage $h(x) = P_x(\tau_z < \tau_a)$.

Connection to the random walk:

Return time $\tau_x^+ := \min\{n \geq 1 : X_n = x\}$.

Lemma: $R_{eff}(a \leftrightarrow z) = \frac{1}{\pi(a) P_a(\tau_z < \tau_a^+)}$

Proof: Let $h(x) = P_x(\tau_z < \tau_a)$

$\pi(a)\mathbb{P}_a(\tau_z<\tau_a)$

Proof: Let $h(x)=\mathbb{P}_x(\tau_z<\tau_a)$.

So that $R_{eff}(a\leftrightarrow z)=\dfrac{1}{\|\theta_h\|}=$

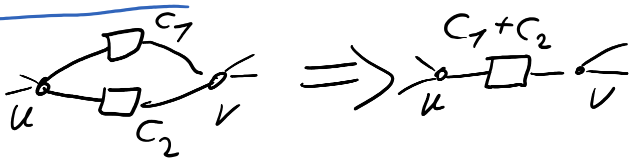
$$=\frac{1}{\sum\limits_{v:v\sim a}\theta_h(av)}=\frac{1}{\sum\limits_{v:v\sim a}c_{av}(h(v)-h(a))}=$$

Total prob. formula

$$=\frac{1}{\sum\limits_{v:v\sim a}c_{av}\mathbb{P}_v(\tau_z<\tau_a)}=\frac{1}{\pi(a)\mathbb{E}[\mathbb{P}_a(\tau_z<\tau_a^+|X_1)]}=$$

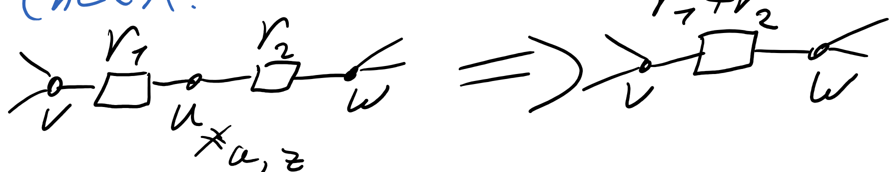
$$=\frac{1}{\pi(a)\mathbb{P}_a(\tau_z<\tau_a^+)}.$$

Network simplifications:

Parallel law: 

If e_1, e_2 are parallel edges, the effective resistance stays if they are replaced by a single edge with the sum of conductances.

Exercise to check.

Series law: 

Exercise

Gluing: If a non-constant voltage h is constant on a subset $S\subseteq V$, then gluing all vertices in S does not change the effective resistance.

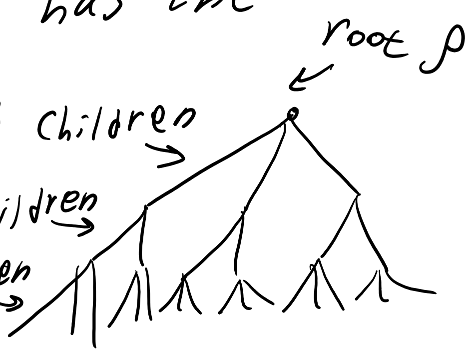
Identify the vertices in S into one vertex and all edges remain (maybe creating parallel edges or self loops).

proof: The same voltage function as on

parallel edges or self loops.

proof: The same voltage function as on the original graph is still a voltage on the glued graph, and has the same strength.

Example: Spherically-symmetric tree →

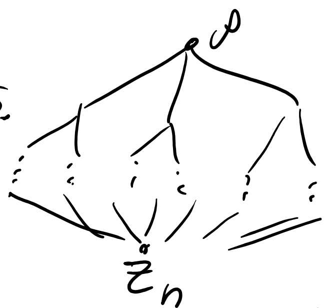


Let T_n be all vertices at level n. Start a simple random walk on the tree from ρ.

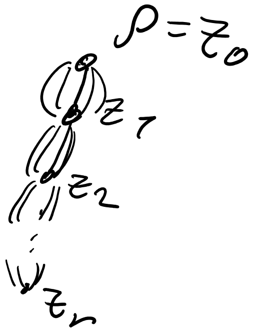
What is $P_\rho(\underbrace{\tau_{T_n}}_{\text{minimal time to reach level } n} < \tau_\rho^+)$

Truncate the tree after level n, and identify all vertices at level n to a single vertex.

Put unit conductances on edges.



The voltage on this graph is constant on each level by symmetry, so we glue the vert. of each level.



Between z_k and z_{k+1} we have $|T_{k+1}| = d_0 \cdot \ldots \cdot d_k$ parallel edges.

By parallel law, these can be replace by single edge with resistance $\frac{1}{|T_{k+1}|}$.

Sum these by series law to get

$$R_{eff}(\rho \leftrightarrow z_n) = \frac{1}{\sum^{n} \frac{1}{|T|}} = \frac{1}{\sum^{n} \prod^{k-1} \frac{1}{}}$$

$$R_{eff}(\rho \leftrightarrow Z_n) = \sum_{k=1}^{n} \frac{1}{|T_k|} = \sum_{k=1}^{n} \prod_{j=1}^{k-1} \frac{1}{d_j}$$

and $\mathbb{P}_\rho(\tau_{T_n} < \tau_\rho^+) = \dfrac{1}{d_\rho R_{eff}(\rho \leftrightarrow Z_n)}$

Note also $\lim_{n\to\infty} \mathbb{P}_\rho(\tau_{T_n} < \tau_\rho^+) = \mathbb{P}_\rho$(random walk never returns to ρ)

$$= \frac{1}{\sum_{k=1}^{\infty} \prod_{j=1}^{k-1} \frac{1}{d_j}}$$

and this says the tree is <u>recurrent</u> iff $\sum_{k=1}^{\infty} \prod_{j=1}^{k-1} \frac{1}{d_j} = \infty$.

The commute time identity

$$\mathbb{E}_a(\tau_z) + \mathbb{E}_z(\tau_a) = 2R_{eff}(a \leftrightarrow z) \sum_{e \in E} c_e.$$

We will not prove this now (exercise).

Energy

We now turn the space of $\theta : \vec{E} \to \mathbb{R}$ to a Hilbert space.

Def: The energy of a flow θ is

$$\mathcal{E}(\theta) := \frac{1}{2} \sum_{\vec{e} \in \vec{E}} r_{\vec{e}}\, \theta(\vec{e})^2 = \sum_{e \in E} r_e \theta(e)^2$$

$\langle \theta_1, \theta_2 \rangle := \frac{1}{2} \sum_{\vec{e} \in \vec{E}} r_{\vec{e}}\, \theta_1(\vec{e})\theta_2(\vec{e})$

Each edge is taken with both orientations

It is generally hard to calculate $R_{eff}(a \leftrightarrow z)$ and difficult to find current flow. Due to this, the following theorem is extremely useful.

William Thomson, Lord Kelvin

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Theorem (Thomson's principle):

$$R_{eff}(a \leftrightarrow z) := \inf\{\mathcal{E}(\theta) : \theta \text{ is a flow from } a \text{ to } z \text{ with } \|\theta\| = 1\}$$

and the unique minimizer is the unit current flow.

Proof: First, we will show that for the unit current flow I it holds that

$$R_{eff}(a \leftrightarrow z) = \mathcal{E}(I).$$

$$\mathcal{E}(I) = \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} r_{xy} I(xy)^2 = \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} r_{xy} c_{xy}(h(y)-h(x)) I_{xy}$$

$I_{xy} = c_{xy}(h(y)-h(x))$

$$= \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} h(y) I_{xy} - \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} h(x) I_{xy} = (*)$$

Node law: $\displaystyle\sum_{x\in V}\sum_{y:y\sim x} h(x) I_{xy} = h(a)\underbrace{\sum_{v:v\sim a} I_{av}}_{=\|I\|=1} + h(z)\underbrace{\sum_{u:u\sim z} I_{zu}}_{=-\|I\|=-1} =$

$$= h(a) - h(z)$$

Similarly, using the antisymmetry,

$$\sum_{x\in V}\sum_{y:y\sim x} h(y) I_{xy} = -(h(a)-h(z))$$

$$\Rightarrow (*) = -(h(a)-h(z)) = \frac{h(z)-h(a)}{\|I\|} = R_{eff}(a \leftrightarrow z)$$

$\|I\| = 1$

We now show that $\mathcal{E}(J) \geq \mathcal{E}(I)$ for every flow J with $\|J\| = 1$.

Set $\theta := J - I$, so that $\|\theta\| = 0$.

$$\mathcal{E}(J) = \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} r_{xy}(I_{xy} + \theta_{xy})^2 =$$

$$= \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} r_{xy}(I_{xy}^2 + \theta_{xy}^2 + 2 I_{xy}\theta_{xy}) =$$

$$\ldots \mathcal{E}(\theta) + \sum \sum r_{xy}\theta_{xy} I_{xy}$$

$$= \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} r_{xy}(I_{xy} \ldots$$

$$= \mathcal{E}(I) + \mathcal{E}(\theta) + \sum_{x\in V}\sum_{y:y\sim x} r_{xy}\theta_{xy}I_{xy}$$

It suffices to show that the last sum is 0.

$$\sum_{x\in V}\sum_{y:y\sim x} r_{xy}\theta_{xy}\, c_{xy}(h(y)-h(x)) =$$

$$= \sum_{x\in V}\sum_{y:y\sim x} \theta_{xy}(h(y)-h(x)) \overset{\text{same calculation as above}}{=} 2(h(z)-h(a))\cdot\|\theta\| = 0.$$

since $\|\theta\| = 0$

Corollary (Rayleigh monotonicity law):

Consider a finite, connected graph $G=(V,E)$ with $a,z\in V$ distinct and two resistances $(r_e)_{e\in E}$, $(r_e')_{e\in E}$. Suppose $r_e' \geq r_e \;\forall e\in E$.

Then $R_{eff}(a\leftrightarrow z; G,(r_e)) \leq R_{eff}(a\leftrightarrow z; G,(r_e'))$

Notice that the limit $r_e' = \infty$ corresponds to removing the edge e and the limit $r_e = 0$ corresponds to gluing the endpoints of e.

Consequently, $\mathbb{P}_a(\tau_z < \tau_a^+)$ cannot increase when removing an edge and cannot decrease when gluing the endpoints of an edge.

Proof: For each flow

$$\mathcal{E}_{(r_e')}(\theta) \geq \mathcal{E}_{(r_e)}(\theta)$$

since $\mathcal{E}_{(r_e)}(\theta) = \sum_{e\in E} r_e\theta(e)^2$.

Now use Thomson's principle.

Corollary: Gluing an arbitrary subset of vertices (without gluing a to z)

Corollary: Gluing ... of vertices (without gluing a to z) cannot increase the effective resistance from a to z.

Proof: Every flow on the original network is still a flow on the glued network with the same strength and energy. Thus the infimum in Thomson's principle cannot increase.

Thomson's principle allows to upper bound the effective resistance. We now give a way to lower bound it.

Def.: The (Dirichlet) energy of a function $h: V \to \mathbb{R}$ is $\mathcal{E}(h) := \sum_{x \sim y} c_{xy} (h(x) - h(y))^2$

(the L^2-norm of the gradient of h with conductances as coefficients).

Theorem (Dirichlet's principle):

$$\frac{1}{R_{eff}(a \leftrightarrow z)} = \inf \{ \mathcal{E}(h) : h: V \to \mathbb{R}, h(a) = 0, h(z) = 1 \}$$

The unique infimum is the voltage with $h(a) = 0$, $h(z) = 1$ $\left(h(x) = P_x(\tau_z < \tau_a) \right)$.

Proof: (Sketch): It is straightforward that the minimal function is harmonic at all vertices except, maybe, a and z.

(Given $h(y)$ for all $y \neq x$, the value $\frac{1}{\pi(x)} \sum_{y: y \sim x} c_{xy} h(y)$ minimizes $\sum_{y: y \sim x} c_{xy} (h(y) - h(x))^2$).

A simple calculation shows that for

$y : y \sim x$

A simple calculation shows that for the voltage with $h(a)=0$, $h(z)=1$,

$$\mathcal{E}(h) = \frac{1}{R_{eff}(a \leftrightarrow z)}.$$

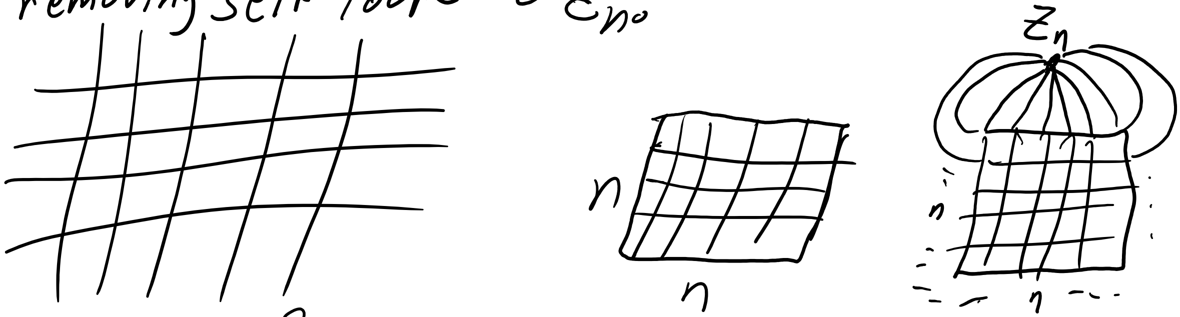
Infinite networks

Let $G=(V,E)$ infinite, connected graph with $C=(C_e)_{e\in E}$ positive (conductances) with $\sum_{y:y\sim x} C_{xy} < \infty$ $\forall x$.

The equality $R_{eff}(a \leftrightarrow z) = \frac{1}{\pi(a)\mathbb{P}_a(\tau_z < \tau_a^+)}$ motivates us to define $R_{eff}(a \leftrightarrow \infty)$ so that we can find $\mathbb{P}_a$(random walk never returns to a)

Let $G_n \subseteq G$ to be finite induced subgraphs which increase to G ($G_n \subseteq G_{n+1}$, $\bigcup G_n = G$).

Define $\overline{G_n}$ by gluing in G all the vertices outside G_n to a single vertex z_n, and removing self loops at z_n.



$G = \mathbb{Z}^2$ $\qquad$ G_n $\qquad$ $\overline{G_n}$

For every $a \in G_1$,

Lemma: $R_{eff}(a \leftrightarrow z_n; \overline{G_n})$ is non-decreasing.

Proof: $\overline{G_n}$ is formed from $\overline{G_{n+1}}$ by gluing.

can define

Proof: G_n is formed from G_{n+1} by gluing.

Therefore we can define

$$\lim_{n\to\infty} R_{eff}(a \leftrightarrow z_n; G_n) =: R_{eff}(a \leftrightarrow \infty; G)$$

This definition does not depend on (G_n) (since we can form an "interlaced" sequence from a pair (G_n) and (G_n')).

Since $\mathbb{P}_a(\underbrace{\tau_a^+ = \infty}_{\text{random walk never returns to } a}) = \lim_{n\to\infty} \mathbb{P}_a(\tau_{z_n} < \tau_a^+) =$

$$= \lim_{n\to\infty} \frac{1}{\pi(a) R_{eff}(a \leftrightarrow z_n; G_n)}$$

$$= \frac{1}{\pi(a) R_{eff}(a \leftrightarrow \infty)}.$$

Conclusion: G is recurrent iff

$$R_{eff}(a \leftrightarrow \infty) = \infty$$

for some (and then every) $a \in V$.

Thomson's principle continues to hold on infinite networks:

$$R_{eff}(a \leftrightarrow \infty) = \inf \left\{ \mathcal{E}(\theta) : \begin{array}{l} \theta \text{ is a flow} \\ \text{from } a \text{ to } \infty \\ \text{with } \|\theta\| = 1 \end{array} \right\}$$

θ is a flow from a to ∞ if it is antisymmetric and the node law holds at all vertices except, maybe, a.

Conclusion: G is transient iff there exists a flow from a to ∞ of finite energy.

exists a flow from

finite energy.